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Polynomial convexity, special polynomial polyhedra and the pluricomplex Green function for a compact set in $\mathbb{C}^n$

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Abstract

We prove a version of the Hilbert Lemniscate Theorem in $\mathbb{C}^n$. More precisely, any polynomially convex compact subset K of $\mathbb{C}^n$ can be approximated externally by special polynomial polyhedra $\mathcal{P}$ defined by proper polynomial mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$ with "almost" all their zeros in $\mathcal{P}$. We precise this version when K is balanced. And we also give several applications of these results: approximation of the pluricomplex Green function for a compact set with pole at infinity, a Lelong–Bremermann Theorem for functions in $\mathcal{L}^+$ and uniform polynomial approximation of holomorphic functions.

Résumé

Nous montrons une version du théorème de Hilbert sur les lemniscates dans $\mathbb{C}^n$. Plus précisément, tout ensemble compact polynomialement convexe K de $\mathbb{C}^n$ est exhausté par des polyèdres polynomiaux spéciaux $\mathcal{P}$ définis par des applications polynomiales propres de $\mathbb{C}^n$ dans $\mathbb{C}^n$ avec « presque » tous leurs zéros dans $\mathcal{P}$. Nous précisons cette version lorsque K est disqué. Et nous donnons aussi plusieurs applications de ces résultats : l'approximation de la fonction de Green pluricomplexe d'un compact avec pôle à l'infini, un théorème de Lelong–Bremermann pour les fonctions dans $\mathcal{L}^+$ et l'approximation uniforme des fonctions holomorphes par des polynômes.

Keywords: Complex analysis; Polynomial convexity; Plurisubharmonic function; Problems of approximation

1. Introduction and statement of results

1.1. Polynomial convexity and special polynomial polyhedra

A compact subset K of $\mathbb{C}^n$ is polynomially convex if, for each $z \in \mathbb{C}^n \setminus K$, there exists a polynomial q such that $|q(z)| > \|q\|_K$. A simple compactness argument then shows that, given an open neighborhood $\mathcal{U}$ of K, there is a finite set of (holomorphic) polynomials $q_1, \dots, q_l$ such that

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$$\sup_{1\leqslant j\leqslant l}\frac{|q_j(z)|}{\|q_j\|_K}>1, \quad z\in\mathbb{C}^n\setminus\mathcal{U}.$$

D. Hilbert proved in 1897, that in fact one polynomial is sufficient in $\mathbb{C}$.

Hilbert Lemniscate Theorem in $\mathbb{C}$ (HLT). *(See [15].) Let K be a polynomially convex compact subset of $\mathbb{C}$ and let $\mathcal{U}$ be an open neighborhood of K. Then there exists a polynomial q such that*

$$\frac{|q(z)|}{\|q\|_K}>1, \quad z\in\mathbb{C}\setminus\mathcal{U}.$$

The polynomial q which appears in this theorem has of course all its zeros in $\mathcal{U}$. A proof of this theorem by classical potential theory (see P. Montel [22], E. Hille [16] and T. Ransford [27]) permits in fact to choose q (Fekete polynomial) such that all its zeros are in K (or even in ∂K). This proof gives in addition the uniform approximation of g_K, the Green function with pole at infinity associated to the compact set K, by subharmonic functions of the type $(\log|q|)/d$ (where d is the degree of the polynomial q) on $\mathbb{C}\setminus\mathcal{U}$.

The term "lemniscate" in the complex plane denotes a curve of the form $L_\varrho := \{z\in\mathbb{C}\colon \prod_{j=1}^m|z-z_j|=\varrho\}$ ($\varrho>0$ and $z_1,\dots,z_m$ are m points in $\mathbb{C}$). The lemniscates L_ϱ can be described (a) as the level curves of polynomials; (b) as the locus of points the product of whose distances from a finite set of fixed points is constant, and (c) as the locus of points $\sum_{j=1}^m\log|z-z_j|=\text{const}$, the left-hand side being a sum of fundamental solutions in $\mathbb{C}$ of Laplace's equation.

In this paper we are interested by the following problem in $\mathbb{C}^n$ for $n\geqslant 2$:

Problem. Approximate a polynomially convex compact subset K in $\mathbb{C}^n$ by polynomial polyhedra $\mathcal{P}$, defined by exactly n polynomials, such that the common zeros of these polynomials are all in $\mathcal{P}$.

A polynomial polyhedron of type N in $\mathbb{C}^n$ is a finite union P of compact connected components of a subset $\tilde{P}$ of the form $\tilde{P}=\{z\in\mathbb{C}^n\colon |p_j(z)|<1 \text{ for } j=1,\dots,N\}$, where p_j are complex polynomials.

A polynomial polyhedron of type n in $\mathbb{C}^n$ is called a *special polynomial polyhedron*.

An equivalent problem is the uniform approximation of g_K, the pluricomplex Green function with pole at infinity associated to the compact set K, by plurisubharmonic functions of the type $\sup_{1\leqslant j\leqslant n}\frac{1}{d_j}\log|p_j|$ (where d_j is the degree of the polynomial p_j) on $\mathbb{C}^n\setminus\mathcal{U}$ (where $\mathcal{U}$ is any neighborhood of K).

Here are three versions, in different cases, of the (HLT) in $\mathbb{C}^n$.

First, consider the case where K is a $\mathcal{L}$-regular polynomially convex compact subset of $\mathbb{C}^n$, i.e. its pluricomplex Green function g_K with pole at infinity is continuous in $\mathbb{C}^n$ (for more details about this subject, see Section 1.2.1).

If the compact set K is not pluripolar, we denote for any positive real number R, $D(R)$ the bounded open sublevel set:

$$D(R)=\bigl\{z\in\mathbb{C}^n\colon g_K(z)<R\bigr\}. \tag{1}$$

Theorem 1 *(The $\mathcal{L}$-regular case). Let K be a $\mathcal{L}$-regular polynomially convex compact subset of $\mathbb{C}^n$. For any $\epsilon>0$ sufficiently small, there exist an integer $d_\epsilon\geqslant 1$ and a proper polynomial mapping $F_\epsilon=(p_1,\dots,p_n)$ of degree d_ϵ such that*

(i) $\|p_j\|_K\leqslant 1$ *for* $\leqslant j\leqslant n$, *and*

$$K\subset\overline{D(\epsilon)}\subset\mathcal{P}_\epsilon\subset D\bigl(\epsilon+\epsilon^2\bigr), \tag{2}$$

where the special polynomial polyhedron $\mathcal{P}_\epsilon$ is the finite union of the connected components of the open set $\{z\in\mathbb{C}^n\colon \sup_{1\leqslant l\leqslant n}\frac{1}{d}\log|p_l(z)|<\epsilon+\beta(\epsilon)\}$ that meet the compact set $\overline{D(\epsilon)}$ and $0<\beta(\epsilon)\leqslant\epsilon^2/2$;

(ii) *the homogeneous component $H_\epsilon:=(p_{1,d_\epsilon},\dots,p_{n,d_\epsilon})$ of degree d_ϵ of F_ϵ (p_{l,d_ϵ} is the homogeneous component of degree d_ϵ of the polynomial p_l) satisfies: $H_\epsilon^{-1}(\{O\})$ is reduced to $\{O\}$;*

(iii) *F_ϵ has a finite set Z_ϵ of zeros in $\mathbb{C}^n$ and each zero is of multiplicity one;*

(iv) *when ϵ tends to 0, d_ϵ tends to infinity, and*

$$\operatorname{card}(Z_\epsilon\cap\mathcal{P}_\epsilon)/d_\epsilon^n\to 1.$$

Remarks. In item (i), the existence of a such special polynomial polyhedron with the "good inclusions" without any precise estimates, is a direct consequence of Bishop's Theorem [5]. Now to obtain precise estimates on the polynomial mapping F_ϵ which defines this polyhedron and in particular to obtain the last estimate about its zeros in item (iv), we need to quantify in details (it is a long and not easy calculation) which appears in the construction of this polynomial mapping. We have already used a similar method in [24] and [25], to prove Zahariuta's Conjecture. Others results about the approximation by special analytic polyhedra can be found in [36].

According to Bezout's Theorem, $d_\epsilon^n = \mathrm{card}(Z_\epsilon)$. Then the last statement (iv) says that "the most part" of zeros of the mapping F_ϵ are in $\mathcal{P}_\epsilon$.

We generalize Theorem 1, when K is a simple polynomially convex compact subset in $\mathbb{C}^n$.

Corollary 2. *Let K be a polynomially convex compact subset of $\mathbb{C}^n$, let $\mathcal{U}$ be an open neighborhood of K and let δ be a positive real number. Then there exists a proper polynomial mapping $F = (p_1, \dots, p_n)$ of degree d such that*

(i) *$K \subset \mathcal{P} \subset \mathcal{U}$, where the special polynomial polyhedron $\mathcal{P}$ is a finite union of connected components of the open neighborhood $\{z \in \mathbb{C}^n\colon \sup_{1\leqslant l\leqslant n} |p_l(z)| < 1\}$ of K*;
(ii) *the homogeneous component $H := (p_{1,d}, \dots, p_{n,d})$ of degree d of F ($p_{l,d}$ is the homogeneous component of degree d of the polynomial p_l) satisfies*: *$H^{-1}(\{O\})$ is reduced to $\{O\}$*;
(iii) *F has a finite set Z of zeros in $\mathbb{C}^n$ and each zero is of multiplicity one*;
(iv) $1 - \delta \leqslant \mathrm{card}(Z \cap \mathcal{P})/d^n \leqslant 1$.

Now, if we consider more carefully the proof of Theorem 1 in the particular case where K is balanced, we obtain the following version of (HLT).

A compact set K in $\mathbb{C}^n$ is balanced if $\bar{\Delta}.K = K$ (Δ is the open unit disc in $\mathbb{C}$).

Theorem 3 *(The balanced case). Let K be a balanced polynomially convex and $\mathcal{L}$-regular compact subset in $\mathbb{C}^n$.*

I *For any $\epsilon > 0$ and $\delta > 0$ sufficiently small, there exist two integers $d \geqslant d' \geqslant 1$ and n polynomials $p_1, \dots, p_n$ of degree d which satisfy the conclusion of Theorem* 1 *and which also verify*:
  (i) *O is a zero of p_j of multiplicity greater or equal to d' for $1 \leqslant j \leqslant n$,*
  (ii) $1 - \delta \leqslant d'/d \leqslant 1$.

II *For any $\epsilon > 0$, there exist a positive integer d and $n+1$ homogeneous polynomials $q_1, \dots, q_{n+1}$ of degree d, with no common zero except the origin, such that $\|q_l\|_K \leqslant 1$ and*

$$K \subset \overline{D(\epsilon)} \subset \mathcal{P} \subset D\big(\epsilon + \epsilon^2\big),$$

where $\mathcal{P} = \{z \in \mathbb{C}^n\colon \sup_{1\leqslant l\leqslant n+1} \frac{1}{d} \log|q_l(z)| < \epsilon + \beta(\epsilon)\}$ and $0 < \beta(\epsilon) \leqslant \epsilon^2/2$.

If we apply Theorem 3-II, in the particular case where K is the closed unit ball in $\mathbb{C}^n$, we obtain an exhaustion of this ball by polynomial polyhedra defined by $\mathbf{n}+1$ homogeneous polynomials. This result improves considerably what we knew until now about this subject. A.B. Aleksandrov [2] proved (by using the homogeneous polynomials introduced by Ryll and Wojtaszczyk [28]) that the smallest number N of homogeneous polynomials which are necessary to define a polynomial polyhedron which exhausts the unit ball, is smaller than 25^{n-1}. It is a consequence of his work about the construction of inner functions [1] (see also Low [21] and Hakim–Sibony [14]).

Since we are not able for the moment to prove in Theorem 1 that all the zeros of the mapping F_ϵ are in $\mathcal{P}_\epsilon$ or to prove in Theorem 3-I that $d' = d$, i.e. the n polynomials p_j are homogeneous of degree d (with unique common zero the origin), it would be natural to solve the following problem in $\mathbb{C}^n$ for $n \geqslant 2$.

Problem. Let $\bar{B}_n(O,1) = \{(z_1, \dots, z_n) \in \mathbb{C}^n\colon \sum_{j=1}^n |z_j|^2 \leqslant 1\}$ be the closed unit Euclidean ball in $\mathbb{C}^n$ for $n \geqslant 1$. For any $\epsilon > 0$, there exist n homogeneous polynomials of degree d ($d \in \mathbf{N}^*$) such that O is their unique common zero and the polynomial polyhedron:

$$\mathcal{P} = \big\{z \in \mathbb{C}^n\colon |p_j(z)| < 1,\ 1 \leqslant j \leqslant n\big\},$$

satisfies,

$$\bar{B}_n(O, 1) \subset \mathcal{P} \subset B_n(O, 1+\epsilon),$$

where $B_n(O, 1+\epsilon)$ is the open Euclidean ball of radius $1+\epsilon$.

This problem is solved for $n = 1$; it is a particular case of (HLT). T. Bloom, N. Levenberg and Yu. Lyubarskii have solved this problem for $n = 2$, in [6]. They reduced it to a problem of one complex variable, by using classical potential theory. Their method is not generalizable in several variables. This problem stays open for $n \geqslant 3$.

1.2. Applications

Pluricomplex Green function for a compact set with pole at infinity

In pluripotential theory, a very interesting question is to approximate a plurisubharmonic function by extremal plurisubharmonic functions with isolated singularities, called pluricomplex Green functions with multiple poles. In $\mathbb{C}^n$, for $n \geqslant 2$, this problem is complicated.

A version with the L^1 approximation in a strongly hyperconvex domain in $\mathbb{C}^n$, for a negative plurisubharmonic function with boundary value zero, has been proved by Poletsky in [26], by using technics developed in [25].

Here we are concerned by this question for plurisubharmonic function in $\mathbb{C}^n$.

A plurisubharmonic function u on $\mathbb{C}^n$ (we also write $u \in PSH(\mathbb{C}^n)$) is of *minimal* (*logarithmic*) *growth* [19] if $u(z) - \log\|z\| \leqslant \emptyset(1)$ as $\|z\| \to \infty$. The family of all such functions is denoted by $\mathcal{L}(\mathbb{C}^n)$ or, simply, $\mathcal{L}$. The family of all functions $u \in PSH(\mathbb{C}^n)$ such that $u(z) - \log\|z\| = \emptyset(1)$ as $\|z\| \to \infty$ is denoted by $\mathcal{L}^+(\mathbb{C}^n)$ or $\mathcal{L}^+$.

Following Zahariuta [35], we define – for any set $E \subset \mathbb{C}^n$ – the function:

$$V_E(z) = \sup\bigl\{u(z){:}\ u \in \mathcal{L}, u \leqslant 0 \text{ on } E\bigr\}, \quad z \in \mathbb{C}^n.$$

It is called the *pluricomplex Green function of the set E* (*with pole at infinity*) (see [4]) or the *$\mathcal{L}$-extremal function of the set E* [31,32].

If K is a compact subset of $\mathbb{C}^n$, then V_K is lower semi-continuous. We denote by g_K the upper regularization V_K^* of V_K.

A set $E \subset \mathbb{C}^n$ is called *$\mathcal{L}$-polar* [19] if there is a function $u \in \mathcal{L}$ such that u is not identically equal to $-\infty$ and $u|_E \equiv -\infty$. According to Siciak [33], if $E \subset \mathbb{C}^n$, then E is $\mathcal{L}$-polar precisely when $g_E = V_E^* \equiv +\infty$. If E is not $\mathcal{L}$-polar, then $g_E \in \mathcal{L}$.

A subset E of $\mathbb{C}^n$ is said to be *$\mathcal{L}$-regular at a point $a \in \bar{E}$* if the pluricomplex Green function V_E is continuous at a. E is said *$\mathcal{L}$-regular* if it is $\mathcal{L}$-regular at each point of $\bar{E}$.

If K is a compact subset of $\mathbb{C}^n$, g_K is maximal in $\mathbb{C}^n \setminus K$ (this notion has been introduced by Sadullaev [29]). According to Bedford and Taylor [8,9], $(dd^c g_K)^n = 0$ in $\mathbb{C}^n \setminus K$ (where $(dd^c.)^n$ is the complex Monge–Ampère operator in $\mathbb{C}^n$).

If K is a non-pluripolar compact subset of $\mathbb{C}^n$, we define the *Siciak extremal function Φ_K* [31] by the following formula:

$$\Phi_K(z) = \sup\bigl\{\bigl|p(z)\bigr|^{1/\deg p}\bigr\}, \quad z \in \mathbb{C}^n,$$

where the supremum is taken over the set of all complex polynomials $p : \mathbb{C}^n \to \mathbb{C}$ such that $\|p\|_K \leqslant 1$ and $\deg(p) \geqslant 1$. It turns out that the plurisubharmonic function $(\log \Phi_K)^*$ behaves, in many ways, like the pluricomplex Green function with pole at infinity.

Zahariuta–Siciak's Theorem. *([35,32,33]) If K is a compact set in $\mathbb{C}^n$, then*

$$V_K = \log \Phi_K.$$

Moreover, $V_K = V_{\hat{K}}$, where $\hat{K}$ denotes the polynomially convex hull of K.

A first application of Theorem 1 is about the approximation of the pluricomplex Green function by maximal plurisubharmonic functions in $\mathcal{L}^+$ with isolated logarithmic poles.

Corollary 4. *Let K be a $\mathcal{L}$-regular polynomially convex compact subset of $\mathbb{C}^n$. For any $\epsilon > 0$, there exist a special polynomial polyhedron $\mathcal{P}_\epsilon$, neighborhood of K (as in Theorem 1), and a function $w_\epsilon \in \mathcal{L}^+$ such that*

(i) *w_ϵ has a finite number of logarithmic poles S_ϵ in $\mathcal{P}_\epsilon$.*
(ii) *w_ϵ is maximal in $\mathbb{C}^n \setminus (\partial\mathcal{P}_\epsilon \cup S_\epsilon)$,*
(iii)
$$g_K(z) - \gamma(\epsilon) \leqslant w_\epsilon(z) \leqslant g_K(z) \quad \text{in } \mathbb{C}^n \setminus \mathcal{P}_\epsilon,$$
where $\gamma(\epsilon) > 0$ tends to 0 when ϵ tends to 0.
(iv) *$\max\{w_\epsilon - \gamma'(\epsilon), 0\} = g_{\overline{\mathcal{P}_\epsilon}}$, in $\mathbb{C}^n$, where $\gamma'(\epsilon) > 0$ tends to 0 when ϵ tends to 0.*
(v) *$\int_{\mathcal{P}_\epsilon} (dd^c w_\epsilon)^n = \int_{S_\epsilon} (dd^c w_\epsilon)^n \to (2\pi)^n = \int_{\mathbb{C}^n} (dd^c w_\epsilon)^n$ and $\int_{\partial\mathcal{P}_\epsilon} (dd^c w_\epsilon)^n \to 0$ when ϵ tends to 0.*

A Lelong–Bremermann's Theorem for functions in $\mathcal{L}^+$

Lelong–Bremermann's Theorem [18], $n = 1$; Bremermann [10], $n \geqslant 1$; and see also Sibony [30]) concerns the representation of a plurisubharmonic function in terms of modulus of holomorphic functions in a pseudoconvex domain in $\mathbb{C}^n$.

Some recent refinements of this theorem are developed in [3].

Here we consider the case where u is a plurisubharmonic function in $\mathbb{C}^n$ contained in $\mathcal{L}^+$. More particularly, we will consider the case where $u = \log h$, with h a plurisubharmonic and non-negative homogeneous (i.e. $h(\lambda z) = |\lambda| h(z)$ for all $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^n$) function in $\mathbb{C}^n$.

Corollary 5.

(i) *Let u be a continuous plurisubharmonic function in $\mathcal{L}^+$. Then for any $\epsilon > 0$ and any compact set L in $\mathbb{C}^n$, there exist $n + 2$ polynomials p_l of degree less or equal to d ($d \in \mathbb{N}^*$) such that*
$$\sup_{1 \leqslant l \leqslant n+2} \frac{1}{d} \log\left|p_l(z)\right| \leqslant u(z) \leqslant \sup_{1 \leqslant l \leqslant n+2} \frac{1}{d} \log\left|p_l(z)\right| + \epsilon, \quad \text{in } L.$$
(ii) *Let h be a plurisubharmonic, continuous, non-negative homogeneous function in $\mathbb{C}^n$, such that $h^{-1}(0) = \{O\}$. Then, for any $\epsilon > 0$, there exist $n + 1$ homogeneous polynomials q_l of degree d ($\geqslant 1$) such that*
$$\sup_{1 \leqslant l \leqslant n+1} \frac{1}{d} \log\left|q_l(z)\right| \leqslant \log h(z) \leqslant \sup_{1 \leqslant l \leqslant n+1} \frac{1}{d} \log\left|q_l(z)\right| + \epsilon, \quad \text{in } \mathbb{C}^n.$$

Uniform polynomial approximation of holomorphic functions

We obtain a precise version of Runge's Theorem, as a consequence of Corollary 2 and by using Weil's integral formula.

Corollary 6. *Let K be a polynomially convex compact subset of $\mathbb{C}^n$ and let f be an holomorphic function in an open neighborhood $\mathcal{U}$ of K. Then there exist a proper polynomial mapping $F = (p_1, \ldots, p_n)$ of degree d in $\mathbb{C}^n$ and a special polynomial polyhedron $\mathcal{P}$ (as in Corollary 2) such that $K \subset \mathcal{P} \subset \overline{\mathcal{P}} \subset \mathcal{U}$ and f can be developed in $\mathcal{P}$ with the following series*:
$$f(z) = \sum_{k_1, \ldots, k_n \geqslant 0} P_{\alpha k_1 \ldots k_n}(z) p_1(z)^{k_1} \ldots p_n(z)^{k_n}.$$
$P_{\alpha k_1 \ldots k_n}$ are complex polynomials, of degree smaller than $n.d$, associated to the connected component $\mathcal{P}_\alpha$ of z in $\mathcal{P}$. This series converges uniformly on K to f.

2. Proofs of Theorem 1 and Corollary 2

2.1. Proofs of items (i), (ii) and (iii) of Theorem 1

In the following, we will use several times Theorem 2.1, which is essentially due to J. Siciak [34].

Theorem 2.1. *Let u be a continuous plurisubharmonic function in $\mathcal{L}^+$.*

(i) *For any $\epsilon > 0$ and for any compact set K in $\mathbb{C}^n$, there exist two integers $d = d(\epsilon) \geqslant 1$ and $N = N(\epsilon) \geqslant 1$ and there exist N polynomials $p_1, \dots, p_N$ of degree less or equal to d such that*

$$u(z) - \epsilon \leqslant \sup_{1 \leqslant l \leqslant N} \frac{1}{d} \log\bigl|p_l(z)\bigr| \leqslant u(z) \quad \text{in } K.$$

(ii) *If in addition $\lim_{\lambda \in \mathbb{C}, |\lambda| \to \infty}(u(\lambda z) - \log|\lambda|)$ exists for any $z \in \mathbb{C}^n \setminus \{O\}$, then the previous approximation is uniform in all $\mathbb{C}^n$.*

Proof. The first part is a consequence of a result of J. Siciak [34]. Let just prove the second part. Since $u \in \mathcal{L}$, the following function:

$$h(\lambda, z) = \begin{cases} |\lambda| \exp u(\lambda^{-1} z), & \text{if } \lambda \in \mathbb{C} \setminus \{0\},\ z \in \mathbb{C}^n, \\ \limsup_{\zeta \in \mathbb{C} \to 0, \zeta \neq 0} |\zeta| \exp u(\zeta^{-1} z), & \text{if } \lambda = 0 \in \mathbb{C},\ z \in \mathbb{C}^n. \end{cases}$$

is plurisubharmonic in $\mathbb{C}^{n+1}$; h is non-negative homogeneous and $\exp(u(z)) = h(1, z)$ for all $z \in \mathbb{C}^n$. With the hypothesis, h is also continuous in $\mathbb{C}^{n+1}$. According to [34] (see also Klimek [17]),

$$h(\lambda, z) = \sup\bigl\{|Q|^{1/\deg Q}(\lambda, z)\bigr\}, \quad (\lambda, z) \in \mathbb{C}^{n+1},$$

where the supremum is taken over all complex homogeneous polynomials Q in $\mathbb{C}^{n+1}$ such that $|Q|^{1/\deg Q} \leqslant h$ in $\mathbb{C}^{n+1}$.

For any $\epsilon > 0$, there exist an integer N and N homogeneous polynomials $Q_1, \dots, Q_N$ in $\mathbb{C}^{n+1}$ such that

$$\sup_{1 \leqslant j \leqslant N} \frac{1}{\deg Q_j} \log|Q_j| \leqslant \log h \leqslant \sup_{1 \leqslant j \leqslant N} \frac{1}{\deg Q_j} \log|Q_j| + \epsilon \quad \text{in } S_{n+1}(0, 1),$$

where $S_{n+1}(0, 1)$ is the unit sphere in $\mathbb{C}^{n+1}$. By homogeneity, we obtain the same estimates for $\log h$ in all $\mathbb{C}^{n+1}$, and then for u in all $\mathbb{C}^n$. The proof is complete. □

Now K is a $\mathcal{L}$-regular compact set in $\mathbb{C}^n$ and ϵ is a real number in $]0, 1[$. We apply the first item of Theorem 2.1 to the pluricomplex Green function g_K, for $\epsilon^2/2$ and on the compact set $L = \bar{D}(2\epsilon)$. We fix ϵ, d, N and the N polynomials $p_1, \dots, p_N$ which appear in this theorem. We denote by v_N the following continuous plurisubharmonic function in $\mathbb{C}^n$:

$$v_N(z) = \sup_{1 \leqslant l \leqslant N} \frac{1}{d} \log\bigl|p_l(z)\bigr|, \tag{3}$$

and for any $r \in \mathbb{R}$, its open sublevel set,

$$P_N(r) = \bigl\{z \in \mathbb{C}^n \colon v_N(z) < r\bigr\}. \tag{4}$$

We obtain the following inclusions:

$$\bar{D}(\epsilon) \subset P_N\bigl(\epsilon + \epsilon^2/2, 1\bigr) \subset D\bigl(\epsilon + \epsilon^2\bigr), \tag{5}$$

where $P_N(\epsilon + \epsilon^2/2, 1)$ is the finite union of the connected components of the open set $P_N(\epsilon + \epsilon^2/2)$ that meet $\overline{D(\epsilon)}$.

Next we decide to abandon the uniform approximation of g_K by a sup of a finite (possibly large) number of plurisubharmonic functions of the type $\frac{1}{d} \log|p|$ (where p is a polynomial of degree d), in order to obtain the exhaustion (2) of the set K with the properties listed in Theorem 1.

To prove precisely inclusions (2), we are inspired by the proof of Bishop's Theorem about special holomorphic polyhedra. We refer to [5], the original paper about this subject (see also [23]). A *holomorphic polyhedron* of type N in an open set D in $\mathbf{C}^n$ is a finite union P of relatively compact connected components of a subset $\tilde{P} \subseteq D$ of the form $\tilde{P} = \{z \in D \colon |f_j(z)| < 1 \text{ for } j = 1, \dots, N\}$, where $f_j \in \mathcal{O}(D)$.

A holomorphic polyhedron of type n in a holomorphically convex open set D in $\mathbb{C}^n$ is called a *special holomorphic polyhedron*.

The existence of special holomorphic polyhedra is a rather nontrivial matter, and the principal existence result is the following:

Bishop's Theorem. *Suppose that D is a holomorphically convex open set in $\mathbb{C}^n$. Then whenever K is a holomorphically convex compact subset of D and U is an open neighborhood of K in D, there is a special holomorphic polyhedron P such that $K \subset P \subseteq U$.*

If $N(\epsilon^2/2) = n$ in Theorem 2.1, according to notation (3) and (4) and to inclusions (5), we have (2) where $\mathcal{P}_\epsilon = P_n(\epsilon + \epsilon^2/2, 1)$ and $\beta(\epsilon) = \epsilon^2/2$.

Now, until the end of Section 2.1, we suppose that this integer $N = N(\epsilon^2/2)$ is greater than n.

The proof first consists to apply a process $(\mathcal{P}_N)$ and then to apply a process $(\mathcal{R}_q)$, successively for $q = N, N-1, \dots, n+1$.

Process $(\mathcal{P}_N)$ consists in modifying a mapping $p = (p_1, \dots, p_N)$ slightly so that the new polynomial mapping $\tilde{p} = (\tilde{p}_1, \dots, \tilde{p}_N)$ satisfies: $(\tilde{p}_1/\tilde{p}_N, \dots, \tilde{p}_{N-1}/\tilde{p}_N) : \mathbb{C}^n \setminus \{\tilde{p}_N = 0\} \to \mathbb{C}^{N-1}$ is locally finite in $\mathbb{C}^n \setminus \{\tilde{p}_N = 0\}$; that is, its level sets consist of just isolated points of $\mathbb{C}^n \setminus \{\tilde{p}_N = 0\}$.

For $q = N, N-1, \dots, n+1$, let suppose that we have a polynomial mapping $\tilde{p} = (\tilde{p}_1, \dots, \tilde{p}_q)$ such that the rational mapping $(\tilde{p}_1/\tilde{p}_q, \dots, \tilde{p}_{q-1}/\tilde{p}_q) : \mathbb{C}^n \setminus \{\tilde{p}_q = 0\} \to \mathbb{C}^{q-1}$ is locally finite and such that $\tilde{p}$ defines a polynomial polyhedron of type q which approximate K externally. Then process $(\mathcal{R}_q)$ consists in choosing an integer ν sufficiently large such that the polynomial mapping $g = (g_1, \dots, g_{q-1})$ in $\mathbb{C}^n$, where $g_j = (\tilde{p}_j)^\nu - (\tilde{p}_q)^\nu$, defines a polynomial polyhedron of type $q-1$ which again approximate K externally.

After $N - n$ such constructions, we obtain a special polynomial polyhedron which satisfies inclusions (2).

Remark 2.2. In the same way, we can also approximate any finite number of sublevel sets $D(R_1), \dots, D(R_s)$ $(0 < R_1 < \cdots < R_s)$ by special polynomial polyhedra defined by the *same n* polynomials:

$$\overline{D\big(R_j - \epsilon^2\big)} \subset D\big(R_j - \epsilon^2/2\big) \subset \mathcal{P}_j \subset D(R_j),$$

where $\mathcal{P}_j$ is the finite union of the connected components of the open set:

$$\left\{z \in \mathbb{C}^n \colon \sup_{1 \leqslant l \leqslant n} \frac{1}{d} \log\big|p_l(z)\big| < R_j - \epsilon^2 + \beta(\epsilon)\right\},$$

that meet $\overline{D(R_j - \epsilon^2)}$. If $D(R_j - \epsilon^2)$ is connected (it is the case when R_j is sufficiently large), then $\mathcal{P}_j$ is the connected component of the open set $\{z \in \mathbb{C}^n \colon \sup_{1 \leqslant l \leqslant n} \frac{1}{d} \log|p_l(z)| < R_j - \epsilon^2 + \beta(\epsilon)\}$ that contains $\overline{D(R_j - \epsilon^2)}$, with $0 < \beta(\epsilon) \leqslant \epsilon^2/2$.

Process $(\mathcal{P}_N)$

We use the same notation as in (3), (4). We have N polynomials p_l of degree d such that $\|p_l\|_K \leqslant 1$ and the function p_N is not identically zero in $\mathbb{C}^n$. So $X_N = \{z \in \mathbb{C}^n \colon p_N(z) = 0\}$ is a proper algebraic subvariety in $\mathbb{C}^n$.

Let denote $F_0 = (p_1, \dots, p_n) : \mathbb{C}^n \to \mathbb{C}^n$ the polynomial mapping defined by the n first polynomials p_l. F_0 has an homogeneous component $H_0 = (p_{1,d}, \dots, p_{n,d})$ of degree d, where $p_{l,d}$ for $1 \leqslant l \leqslant n$ is the homogeneous component of degree d of the polynomial p_l. According to a density argument (Lemma 1, p. 431 of [20]: *Let $d_1, \dots, d_n \in \mathbb{N}^*$, and let E be the finite-dimensional vector space of all the mappings $f = (f_1, \dots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$, where f_i is an homogeneous polynomial of degree d_i $(i = 1, \dots, n)$. Then the set $\Sigma = \{f \in E \colon f^{-1}(O) \neq \{O\}\} \subset E$ is algebraic and nowhere dense.*), we can suppose that $H_0^{-1}(\{O\})$ is reduced to $\{O\}$.

Then for any integer $\alpha \geqslant 1$, the polynomials $\tilde{p}_l$ defined by,

$$\tilde{p}_l = p_l^{\alpha+1}, \quad \text{for } 1 \leqslant l \leqslant n,$$
$$\tilde{p}_l = p_l^{\alpha}, \quad \text{for } n+1 \leqslant l \leqslant N,$$

verify that the rational mapping $(\tilde{p}_1/\tilde{p}_N, \dots, \tilde{p}_{N-1}/\tilde{p}_N)$: $\mathbb{C}^n \setminus \{\tilde{p}_N = 0\} \to \mathbb{C}^{N-1}$ is locally finite. Indeed, let $(a_1, \dots, a_{N-1}) \in \mathbb{C}^{N-1}$ be fixed. We denote by $F_{a_1,\dots,a_n}$, the following polynomial mapping of degree $d(\alpha+1)$:

$$F_{a_1,\dots,a_n} = \big(p_1^{\alpha+1} - a_1 p_N^{\alpha}, \dots, p_n^{\alpha+1} - a_n p_N^{\alpha}\big),$$

and by $H_{a_1,\dots,a_n}$ its homogeneous part of degree $d(\alpha+1)$. Since $H^{-1}_{a_1,\dots,a_n}(\{0\})$ is reduced to $\{0\}$, we can easily deduce that $F^{-1}_{a_1,\dots,a_n}(\{0\})$ is a finite set in $\mathbb{C}^n$ (see Lemma 3, p. 433 of [20]). Then the sets,

$$\begin{cases} p_l^{\alpha+1} - a_l p_N^\alpha = 0, & 1 \leqslant l \leqslant n, \\ p_l^\alpha - a_l p_N^\alpha = 0, & n+1 \leqslant l \leqslant N-1, \end{cases} \quad \text{and} \quad \begin{cases} p_l^{\alpha+1}/p_N^\alpha = a_l, & 1 \leqslant l \leqslant n, \\ p_l^\alpha/p_N^\alpha = a_l, & n+1 \leqslant l \leqslant N-1, \\ p_N \neq 0, \end{cases}$$

are finite.

Now let $\tilde{v}_N$ be the continuous plurisubharmonic function in $\mathbb{C}^n$ given by,

$$\tilde{v}_N = \sup_{1\leqslant l\leqslant N} \frac{1}{d(\alpha+1)} \log|\tilde{p}_l|,$$

and for any $r \in \mathbb{R}$, its open sublevel set:

$$\tilde{P}_N(r) = \{z \in \mathbb{C}^n:\ \tilde{v}_N(z) < r\}.$$

For α sufficiently large, we obtain the following inclusions:

$$K \subset \overline{D(\epsilon)} \subset \tilde{P}_N\big(\epsilon + \epsilon^2/2^2, 1\big) \subset D\big(\epsilon + \epsilon^2\big),$$

where $\tilde{P}_N(\epsilon + \epsilon^2/2^2, 1)$ is the finite union of the connected components of the open set $\tilde{P}_N(\epsilon + \epsilon^2/2^2)$ that meet $\overline{D(\epsilon)}$. Indeed, since $\|\tilde{p}_l\|_K \leqslant 1$ for $1 \leqslant l \leqslant N$, $\tilde{v}_N \leqslant g_K$ in $\mathbb{C}^n$ and we have the second inclusion. In addition on the compact set $\overline{D(2\epsilon)}$, we have $\sup_{n+1\leqslant l\leqslant N} \frac{1}{d}\log|p_l| - 2\epsilon \leqslant 0$. Then

$$\sup_{n+1\leqslant l\leqslant N} \frac{1}{d}\log|p_l| - 2\epsilon \leqslant \frac{\alpha}{\alpha+1}\left(\sup_{n+1\leqslant l\leqslant N} \frac{1}{d}\log|p_l| - 2\epsilon\right),$$

and

$$\frac{\alpha}{\alpha+1} \sup_{n+1\leqslant l\leqslant N} \frac{1}{d}\log|p_l| \geqslant \sup_{n+1\leqslant l\leqslant N} \frac{1}{d}\log|p_l| - \frac{2\epsilon}{\alpha+1}.$$

Consequently, $\tilde{v}_N \geqslant v_N - \frac{2\epsilon}{\alpha+1}$ in $\bar{D}(2\epsilon)$ and according to (5), we obtain the third inclusion if $\alpha + 1 > 8/\epsilon$.

We note $d_N = d(\alpha+1)$.

Process ($\mathcal{R}_q$) for $n+1 \leqslant q \leqslant N$

Let suppose that we have $\tilde{v}_q$ a continuous plurisubharmonic function in $\mathbb{C}^n$ given by,

$$\tilde{v}_q = \sup_{1\leqslant l\leqslant q} \frac{1}{d_q} \log|\tilde{p}_l|,$$

where the q polynomials $\tilde{p}_l$, of degree d_q, satisfy:

(i) $\|\tilde{p}_l\|_K \leqslant 1$ for any $1 \leqslant l \leqslant q$.

(ii) The following rational mapping is locally finite:

$$\left(\frac{\tilde{p}_1}{\tilde{p}_q}, \dots, \frac{\tilde{p}_{q-1}}{\tilde{p}_q}\right):\ \mathbb{C}^n \setminus \{\tilde{p}_q = 0\} \to \mathbb{C}^{q-1}. \tag{6}$$

For any $r \in \mathbb{R}$,

$$\tilde{P}_q(r) = \{z \in \mathbb{C}^n:\ \tilde{v}_q(z) < r\}.$$

We suppose also that we have the following inclusions,

$$K \subset \overline{D(\epsilon)} \subset \tilde{P}_q\big(\epsilon + \epsilon^2/2^{2+2(N-q)}, 1\big) \subset D\big(\epsilon + \epsilon^2\big), \tag{7}$$

where $\tilde{P}_q(\epsilon + \epsilon^2/2^{2+2(N-q)}, 1)$ is the finite union of the connected components of the open set $\tilde{P}_q(\epsilon + \epsilon^2/2^{2+2(N-q)})$ that meet $\overline{D(\epsilon)}$.

Now let us denote, for any $1 \leqslant l \leqslant q-1$ and for any $\nu \in \mathbb{N}^*$, the polynomial

$$F_{l,\nu} = \tilde{p}_l^\nu - \tilde{p}_q^\nu.$$

For any $r \in \mathbb{R}$, we denote by $P_{q-1}^\nu(r)$ the open sublevel set:

$$P_{q-1}^\nu(r) = \left\{ z \in \mathbb{C}^n \colon\ w_{q-1}^\nu(z) < r \right\},$$

where $w_{q-1}^\nu = \sup_{1\leqslant l\leqslant q-1} \frac{1}{d_q \nu} \log|F_{l,\nu}|$ is a continuous plurisubharmonic function in $\mathbb{C}^n$.

The following proposition is similar to Proposition 3.3, p. 432 of [25].

Proposition 2.3. *For any $\epsilon > 0$ sufficiently small, if $\alpha_1(\epsilon) = \epsilon^2/2^{2+2(N-q)}$ and $\alpha_2(\epsilon) = \epsilon^2/2^{2(N-q)+3}$, there exists an integer $\nu_q \geqslant 1$ such that for any $\nu \geqslant \nu_q$, we have*:

$$K \subset \overline{D(\epsilon)} \subset P_{q-1}^\nu\big(\epsilon + \alpha_2(\epsilon), 1\big) \subset \tilde{P}_q\big(\epsilon + \alpha_1(\epsilon), 1\big) \subset D\big(\epsilon + \epsilon^2\big), \tag{8}$$

where $P_{q-1}^\nu(\epsilon + \alpha_2(\epsilon), 1)$ is the finite union of the connected components of $P_{q-1}^\nu(\epsilon + \alpha_2(\epsilon))$ that meet $\overline{D(\epsilon)}$.

Proof. For any $l \in \{1, \dots, q-1\}$ and for any $\nu \geqslant 1$, $F_{l,\nu}$ is a polynomial which satisfies $|F_{l,\nu}(z)| \leqslant 2 \sup_{1\leqslant l\leqslant q} |\tilde{p}_l(z)|^\nu$ in $\mathbb{C}^n$. Consequently, $\|F_{l,\nu}\|_K \leqslant 2$ and

$$\begin{aligned} \sup_{1\leqslant l\leqslant q-1} \frac{1}{d_q \nu} \log\big|F_{l,\nu}(z)\big| &\leqslant \sup_{1\leqslant l\leqslant q} \frac{1}{d_q} \log\big|\tilde{p}_l(z)\big| + \frac{\log 2}{d_q \nu} \quad \text{in } \mathbb{C}^n \\ &\leqslant g_K(z) + \frac{\log 2}{d_q \nu} \quad \text{in } \mathbb{C}^n. \end{aligned}$$

Since $K \subset \overline{D(\epsilon)} \subset \tilde{P}_q(\epsilon + \alpha_1(\epsilon), 1) \subset D(\epsilon + \epsilon^2)$; if we choose an integer ν_q^0 such that $\log 2/(d_q \nu_q^0) < \alpha_2(\epsilon)$, then we obtain for $\nu \geqslant \nu_q^0$:

$$\overline{D(\epsilon)} \subset P_{q-1}^\nu\big(\epsilon + \alpha_2(\epsilon)\big) \quad \text{and} \quad \overline{D(\epsilon)} \subset P_{q-1}^\nu\big(\epsilon + \alpha_2(\epsilon), 1\big).$$

To complete the proof of relation (8), it is enough to show that whenever ν is sufficiently large,

$$P_{q-1}^\nu\big(\epsilon + \alpha_2(\epsilon), 1\big) \subset \tilde{P}_q\big(\epsilon + \alpha_1(\epsilon)\big).$$

If it was not the case, then there would be a sequence $(\nu_k)_k$ of integers such that $\nu_k \to \infty$ and

$$P_{q-1}^{\nu_k}\big(\epsilon + \alpha_2(\epsilon), 1\big) \not\subset \tilde{P}_q\big(\epsilon + \alpha_1(\epsilon)\big).$$

In what follows, we will consider integers k sufficiently large (and the corresponding ν_k) such that

$$K \subset \overline{D(\epsilon)} \subset P_{q-1}^{\nu_k}\big(\epsilon + \alpha_2(\epsilon), 1\big).$$

For each connected component $P_{q-1}^{\nu_k,0}(\epsilon + \alpha_2(\epsilon), 1)$ of $P_{q-1}^{\nu_k}(\epsilon + \alpha_2(\epsilon), 1)$, necessarily $P_{q-1}^{\nu_k,0}(\epsilon + \alpha_2(\epsilon), 1) \cap \overline{D(\epsilon)} \neq \emptyset$, and there must be some connected component $P_{q-1}^{\nu_k,0}(\epsilon + \alpha_2(\epsilon), 1)$ for which

$$P_{q-1}^{\nu_k,0}\big(\epsilon + \alpha_2(\epsilon), 1\big) \cap \partial \tilde{P}_q\big(\epsilon + \alpha_1(\epsilon)\big) \neq \emptyset.$$

Let choose a positive constant c_2 such that $0 < \alpha_2(\epsilon) < c_2 < \alpha_1(\epsilon)$ and introduce the auxiliary open level set $\tilde{P}_q(\epsilon + c_2)$ for which $\overline{\tilde{P}_q(\epsilon + \alpha_1(\epsilon))} \setminus \tilde{P}_q(\epsilon + c_2)$ is compact in $\mathbb{C}^n$. We note that there must be some connected component:

$$R_k \subset P_{q-1}^{\nu_k,0}\big(\epsilon + \alpha_2(\epsilon), 1\big) \cap \big[\overline{\tilde{P}_q\big(\epsilon + \alpha_1(\epsilon)\big)} \setminus \tilde{P}_q(\epsilon + c_2)\big],$$

for which

$$R_k \cap \partial \tilde{P}_q\big(\epsilon + \alpha_1(\epsilon)\big) \neq \emptyset \quad \text{and} \quad R_k \cap \partial \tilde{P}_q(\epsilon + c_2) \neq \emptyset.$$

Indeed, choose a path in $P^{\nu_k,0}_{q-1}(\epsilon+\alpha_2(\epsilon),1)$ from a point in $P^{\nu_k,0}_{q-1}(\epsilon+\alpha_2(\epsilon),1)\cap\overline{D(\epsilon)}$ to a point in $P^{\nu_k,0}_{q-1}(\epsilon+\alpha_2(\epsilon),1)\cap\partial\tilde{P}_q(\epsilon+\alpha_1(\epsilon))\neq\emptyset$. Then observe that the segment of that path from the last point in $\overline{\tilde{P}_q(\epsilon+c_2)}$ to the first point in $\partial\tilde{P}_q(\epsilon+\alpha_1(\epsilon))$ belongs to such a connected component R_k. If $z\in R_k$, then $z\in P^{\nu_k}_{q-1}(\epsilon+\alpha_2(\epsilon))$,

$$\left|\tilde{p}_l(z)^{\nu_k}-\tilde{p}_q(z)^{\nu_k}\right|<e^{d_q\nu_k(\epsilon+\alpha_2(\epsilon))},\quad \text{for } l=1,\dots,q-1,$$

and since $z\notin\tilde{P}_q(\epsilon+c_2)$ necessarily:

$$\left|\tilde{p}_{l_0}(z)\right|\geqslant e^{d_q(\epsilon+c_2)}\quad \text{for some index } 1\leqslant l_0\leqslant q.$$

Combining these last two inequalities, we obtain a third relation:

$$\left|\tilde{p}_q(z)\right|^{\nu_k}\geqslant\left|\tilde{p}_{l_0}(z)\right|^{\nu_k}-\left|\tilde{p}_q(z)^{\nu_k}-\tilde{p}_{l_0}(z)^{\nu_k}\right|>e^{d_q\nu_k(\epsilon+c_2)}-e^{d_q\nu_k(\epsilon+\alpha_2(\epsilon))}>0, \tag{9}$$

from which it follows that $R_k\subset\mathbb{C}^n\setminus\{\tilde{p}_q=0\}$, and hence the mapping,

$$\left(\frac{\tilde{p}_1}{\tilde{p}_q},\dots,\frac{\tilde{p}_{q-1}}{\tilde{p}_q}\right):\mathbb{C}^n\setminus\{\tilde{p}_q=0\}\to\mathbb{C}^{q-1},$$

is well defined on R_k. Then, combining the above inequalities yields the following inequality:

$$\left|\frac{\tilde{p}_l(z)^{\nu_k}}{\tilde{p}_q(z)^{\nu_k}}-1\right|=\left|\frac{\tilde{p}_l(z)^{\nu_k}-\tilde{p}_q(z)^{\nu_k}}{\tilde{p}_q(z)^{\nu_k}}\right|<\frac{e^{d_q\nu_k(\epsilon+\alpha_2(\epsilon))}}{e^{d_q\nu_k(\epsilon+c_2)}-e^{d_q\nu_k(\epsilon+\alpha_2(\epsilon))}}=\left(e^{d_q\nu_k(c_2-\alpha_2(\epsilon))}-1\right)^{-1},$$

and since $e^{d_q(c_2-\alpha_2(\epsilon))}>1$, it follows from this that

$$\left|\frac{\tilde{p}_l(z)^{\nu_k}}{\tilde{p}_q(z)^{\nu_k}}-1\right|<\frac{\pi}{\nu_k}\quad \text{for } \nu_k \text{ sufficiently large.}$$

Geometrically, this last inequality means that the point $\tilde{p}_l(z)/\tilde{p}_q(z)$ lies within one of ν_k disjoint open neighborhoods of the ν_k roots of unity, where these neighborhoods have the property that their radii tend to zero as ν_k increases to infinity.

Now, since R_k is connected, the points $\tilde{p}_l(z)/\tilde{p}_q(z)$ must indeed lie in the same neighborhood for all $z\in R_k$. The mapping $z\in R_k\to(\frac{\tilde{p}_1(z)}{\tilde{p}_q(z)},\dots,\frac{\tilde{p}_{q-1}(z)}{\tilde{p}_q(z)})$ thus takes R_k into a product of $q-1$ such neighborhoods. After passing to a suitable subsequence of the indices ν_k if necessary, it can be assumed that these neighborhoods shrink to a single point $(\xi_1,\dots,\xi_{q-1})$, where of course $|\xi_l|=1$. Thus for any points $z_k\in R_k$,

$$\lim_{k\to\infty}\frac{\tilde{p}_l(z_k)}{\tilde{p}_q(z_k)}=\xi_l,\quad 1\leqslant l\leqslant q-1.$$

Now for any value t in the interval $[c_2,\alpha_1(\epsilon)]$, there must be some point $z^k_t\in R_k\subset\overline{\tilde{P}_q(\epsilon+\alpha_1(\epsilon))}\setminus\tilde{P}_q(\epsilon+c_2)$ for which $\sup_l|\tilde{p}_l(z^k_t)|=e^{d_q(\epsilon+t)}$, and since $\overline{\tilde{P}_q(\epsilon+\alpha_1(\epsilon))}\setminus\tilde{P}_q(\epsilon+c_2)\subset\overline{\tilde{P}_q(\epsilon+\alpha_1(\epsilon))}$, which is a compact set in $\mathbb{C}^n$, a subsequence of these points will converge to a limit point $z_t\in\overline{\tilde{P}_q(\epsilon+\alpha_1(\epsilon))}\setminus\tilde{P}_q(\epsilon+c_2)$. Since $\sup_l|\tilde{p}_l(z_t)|=e^{d_q(\epsilon+t)}$, these points are distinct for distinct values of t, so there are indeed uncountably many such points.

The values $\tilde{p}_q(z^k_t)$ are uniformly bounded away from zero as a consequence of (9). Indeed,

$$\left|\tilde{p}_q\left(z^k_t\right)\right|\geqslant e^{d_q(\epsilon+c_2)}\left(1-e^{d_q\nu_k(\alpha_2(\epsilon)-c_2)}\right)^{1/\nu_k}=e^{d_q(\epsilon+c_2)}\exp\left(\frac{1}{\nu_k}\log\left(1-e^{d_q\nu_k(\alpha_2(\epsilon)-c_2)}\right)\right),$$

where the last factor $\exp(\frac{1}{\nu_k}\log(1-e^{d_q\nu_k(\alpha_2(\epsilon)-c_2)}))$ tends to 1 when k tends to ∞. Thus $|\tilde{p}_q(z_t)|\neq 0$. It then follows that all the points z_t have the same image under the mapping (6), contradicting the condition that the mapping is locally finite and hence can have at most countably many inverse images. That contradiction means that it must be the case that for $\alpha_2(\epsilon)$ chosen as before, $P^{\nu}_{q-1}(\epsilon+\alpha_2(\epsilon),1)\subset\tilde{P}_q(\epsilon+\alpha_1(\epsilon))$ whenever ν is sufficiently large (i.e. for $\nu\geqslant\nu_q\geqslant\nu^0_q$). In addition, each connected component of $P^{\nu}_{q-1}(\epsilon+\alpha_2(\epsilon),1)$ is contained in a connected component of $\tilde{P}_q(\epsilon+\alpha_1(\epsilon))$, which then itself meets $\overline{D(\epsilon)}$. Consequently, the inclusions (8) are proved and the proof of Proposition 2.3 is complete. □

Process $(\mathcal{R}_q)$ consists in applying Proposition 2.3 to the $q-1$ polynomials $F_{l,\nu}$ defined above.

Then we choose ν sufficiently large such that $\log 2/(d_q\nu) \leqslant \epsilon^2/2^{2(N-q)+4}$. We define $q-1$ new polynomials $\tilde{p}_l$ by $F_{l,\nu}/2$. They are of degree $d_{q-1} = d_q\nu$ and $\|\tilde{p}_l\|_K \leqslant 1$ for any $1 \leqslant l \leqslant q-1$. We also denote by $\tilde{P}_{q-1}(r)$ the new open sublevel set

$$\tilde{P}_{q-1}(r) = \{z \in \mathbb{C}^n \colon \tilde{v}_{q-1}(z) < r\},$$

where $\tilde{v}_{q-1}$ is the plurisubharmonic function in $\mathbb{C}^n$ defined by:

$$\tilde{v}_{q-1} = \sup_{1\leqslant l\leqslant q-1} \frac{1}{d_q\nu} \log|\tilde{p}_l|.$$

Remark that $\tilde{v}_{q-1} \leqslant g_K$ in $\mathbb{C}^n$. According to (8), we obtain inclusions (7) with $q-1$. In addition, the new rational mapping (6) for $q-1$ is locally finite. Indeed, it is sufficient to remark that the homogeneous part of maximal degree d_{q-1} of the n first coordinates $(\tilde{p}_1, \ldots, \tilde{p}_n)$ of this mapping $(\tilde{p}_1, \ldots, \tilde{p}_{q-1})$, has an unique zero at the origin.

Conclusion: iteration of the process $(\mathcal{R}_q)$ successively for $q = N, N-1, \ldots, n+1$

We use the same notation as after process $(\mathcal{P}_N)$. Therefore, we apply process $(\mathcal{R}_N)$ to obtain the existence of $N-1$ new polynomials $\tilde{p}_l$ of degree d_{N-1} ($d_{N-1} \geqslant d_N$) which satisfy all the conditions to apply process $(\mathcal{R}_{N-1})$ again. It is clear now that, by iterating process $(\mathcal{R}_q)$ for $q = N-1$, for $q = N-2, \ldots$ and finally for $q = n+1$, we obtain the existence of n new polynomials p_l of degree d_n ($d_n \geqslant d_N$ and d_n is d_ϵ in Theorem 1) such that

(i) $\|p_l\|_K \leqslant 1$ for $1 \leqslant l \leqslant n$,
(ii) the inclusions (2) in Theorem 1 are satisfy with $\mathcal{P}_\epsilon = \tilde{P}_n(\epsilon + \beta(\epsilon), 1)$ and $\beta(\epsilon) = \epsilon^2/2^{2(N-n+1)}$,
(iii) the mapping $F_\epsilon = (p_1, \ldots, p_n)$ satisfies items (ii) and (iii) of Theorem 1.

Indeed F_ϵ has an homogeneous component $H_\epsilon = (p_{1,d_\epsilon}, \ldots, p_{n,d_\epsilon})$ of degree d_ϵ, where p_{l,d_ϵ} for $1 \leqslant l \leqslant n$ is the homogeneous component of degree d_ϵ of the polynomial p_l. p_{l,d_ϵ} is the power $d_\epsilon/(d(\alpha+1))$ of the homogeneous component of degree $d(\alpha+1)$ of the polynomial $\tilde{p}_l$, which appears in process $\mathcal{P}_N$.

And by assumption, $H_\epsilon^{-1}(\{O\})$ is reduced to $\{O\}$. Then F_ϵ has a finite set of zeros in $\mathbb{C}^n$ (Lemma 3, p. 433 of [20]). In addition, we can suppose that they are all of multiplicity equal to 1, if we replace F_ϵ by $F_\epsilon - a$ where $a \in \mathbb{C}^n$ is a regular value of F_ϵ near O.

2.2. Proof of the item (iv) of Theorem 1

Proper polynomial mappings

In this paragraph, we use the same notation as in the previous paragraph 2.1. We denote by v_ϵ the following plurisubharmonic function in $\mathbb{C}^n$:

$$v_\epsilon(z) := \sup_{1\leqslant l\leqslant n} \frac{1}{d_\epsilon} \log|p_l(z)|, \tag{10}$$

where $\beta(\epsilon) = \epsilon^2/2^{2(N-n+1)}$ and $r_1 = \exp[d_\epsilon(\epsilon + \beta(\epsilon))]$. We call P_1 the polydisc in $\mathbb{C}^n$ centered in O with multi-radius $r_1.(1, \ldots, 1)$. F_ϵ is the polynomial mapping,

$$F_\epsilon = (p_1, \ldots, p_n) : \mathbb{C}^n \to \mathbb{C}^n,$$

with finite zero set Z_ϵ in $\mathbb{C}^n$.

Proposition 2.4. *The mapping F_ϵ is proper and surjective from $\mathbb{C}^n$ to $\mathbb{C}^n$ and from the bounded special polynomial polyhedron $\mathcal{P}_\epsilon$ to the polydisc P_1.*

Proof. Since $H_\epsilon^{-1}(\{O\})$ is reduced to $\{O\}$, there exist positive constants C_0 and R_0 such that $\|F_\epsilon(z)\| \geqslant C_0\|z\|^{d_\epsilon}$ for $\|z\| \geqslant R_0$. Then $\lim_{\|z\|\to\infty} \|F_\epsilon(z)\| = +\infty$ and the mapping $F_\epsilon : \mathbb{C}^n \to \mathbb{C}^n$ is proper.

According to Remmert's proper mapping theorem ([20], pp. 290, 300), since $F:\mathbb{C}^n \to \mathbb{C}^n$ is proper, F_ϵ is surjective.

The fact that F_ϵ is proper from $\mathcal{P}_\epsilon$ to P_1 directly comes from the definition of $\mathcal{P}_\epsilon$. $\square$

According to Bezout's Theorem and to the fact that all the multiplicities of zeros are equal to 1, the cardinal of Z_ϵ is equal to d_ϵ^n. We denote by k this cardinal, $Z_\epsilon = \{\alpha_1, \dots, \alpha_k\}$ and $Z'_\epsilon := \{\alpha_1, \dots, \alpha_{k'}\}$ (where $1 \leqslant k' \leqslant k$) is the zero set of F_ϵ in $\mathcal{P}_\epsilon$. Z'_ϵ is not empty according to Proposition 2.4 and F_ϵ has no zero in $\partial\mathcal{P}_\epsilon$.

Where are the zeros of the mapping F_ϵ?

The following lemma is a special case of a more general result proved by Demailly [12,13]. It will be useful for proving Lemma 2.6.

Lemma 2.5 *("Comparison Theorem"). Let D be a bounded hyperconvex domain in $\mathbb{C}^n$. Let $u, v : D \to [-\infty, 0[$ be two continuous exhaustion functions in D such that $u \leqslant v \leqslant 0$ in $U \cap D$, where U is a neighborhood of ∂D. Then*

$$\int_D \left(dd^c v\right)^n \leqslant \int_D \left(dd^c u\right)^n.$$

Let us consider the finite set $Z'_{\epsilon,o} = \{(\alpha_j, 1/d_\epsilon, 1 \leqslant j \leqslant k'\}$ in $\mathcal{P}_\epsilon$ of poles α_j associated respectively with the weights $1/d_\epsilon$. Let denote v_1, the pluricomplex Green function $g_{D(\epsilon+\epsilon^{3/2})}(Z'_{o,\epsilon}, .)$ in $D(\epsilon+\epsilon^{3/2})$ with logarithmic poles in $Z'_{o,\epsilon}$.

Lemma 2.6.

$$\int_{D(\epsilon+\epsilon^{3/2})} \left(dd^c v_1\right)^n \to (2\pi)^n \quad \textit{when } \epsilon \to 0,$$

which is equivalent to say that

$$\frac{\operatorname{card}(Z_\epsilon \cap \mathcal{P}_\epsilon)}{d_\epsilon^n} = \frac{\operatorname{card} Z'_\epsilon}{d_\epsilon^n} \to 1 \quad \textit{when } \epsilon \to 0.$$

Proof. Let recall that $v_\epsilon = \sup_{1 \leqslant l \leqslant n} \frac{1}{d_\epsilon} \log|p_l| \leqslant g_K$ on $\mathbb{C}^n$ (v_ϵ is defined in (10)). Then v_ϵ satisfies: $v_\epsilon - \epsilon - \epsilon^{3/2} \leqslant 0$ on $D(\epsilon+\epsilon^{3/2})$, and in particular on $\overline{\mathcal{P}_\epsilon}$. By definition of v_1, we have $v_1 \geqslant v_\epsilon - \epsilon - \epsilon^{3/2}$ on $D(\epsilon+\epsilon^{3/2})$ and in particular $v_1 \geqslant \beta(\epsilon) - \epsilon^{3/2}$ on $\partial\mathcal{P}_\epsilon$. $v_1 = 0$ on $\partial D(\epsilon+\epsilon^{3/2})$. Also,

$$g_K - \left(\epsilon+\epsilon^{3/2}\right) \leqslant \epsilon^2 - \epsilon^{3/2} \quad \text{on } \partial\mathcal{P}_\epsilon \quad \text{and} \quad g_K - \left(\epsilon+\epsilon^{3/2}\right) = 0 \quad \text{on } \partial D\left(\epsilon+\epsilon^{3/2}\right).$$

Since v_1 is maximal on $D(\epsilon+\epsilon^{3/2}) \setminus \overline{\mathcal{P}_\epsilon}$, we obtain:

$$\begin{aligned} v_1 &\geqslant \frac{(g_K - (\epsilon+\epsilon^{3/2}))(\epsilon^{3/2} - \beta(\epsilon))}{-\epsilon^2 + \epsilon^{3/2}} \\ &\geqslant \delta_\epsilon\left(g_K - \left(\epsilon+\epsilon^{3/2}\right)\right) \quad \text{on } D\left(\epsilon+\epsilon^{3/2}\right) \setminus \overline{\mathcal{P}_\epsilon}, \end{aligned}$$

where $\delta_\epsilon := \frac{\epsilon^{3/2} - \beta(\epsilon)}{\epsilon^{3/2} - \epsilon^2}$ tends to 1 when ϵ tends to 0. By applying Lemma 2.5, we deduce that

$$\begin{aligned} \int_{D(\epsilon+\epsilon^{3/2})} \left(dd^c v_1\right)^n &\leqslant \delta_\epsilon^n \int_{D(\epsilon+\epsilon^{3/2})} \left(dd^c g_K\right)^n \\ &= \delta_\epsilon^n \int_{\mathbb{C}^n} \left(dd^c g_K\right)^n = \delta_\epsilon^n (2\pi)^n, \end{aligned}$$

where the last equalities arise from the fact that g_K is maximal on $\mathbb{C}^n \setminus K$ and is in $\mathcal{L}^+$.

On the other hand, $v_1 \leqslant g_{\mathcal{P}}(Z'_{\epsilon,o}, .) = v_\epsilon - (\epsilon + \beta(\epsilon))$ on $\mathcal{P}_\epsilon$, because $\mathcal{P}_\epsilon \subset D(\epsilon+\epsilon^2) \subset D(\epsilon+\epsilon^{3/2})$. Thus we have on $\partial D(\epsilon^2) \subset D(\epsilon) \subset \mathcal{P}_\epsilon$,

$$v_1 \leqslant -\epsilon + \epsilon^2 - \beta(\epsilon),$$

and

$$g_K - \big(\epsilon + \epsilon^{3/2}\big) = \epsilon^2 - \epsilon - \epsilon^{3/2}.$$

Consequently, according to the maximality of g_K on $\mathbb{C}^n \setminus K$, we deduce that

$$v_1 \leqslant \frac{(g_K - (\epsilon + \epsilon^{3/2}))(\epsilon - \epsilon^2 + \beta(\epsilon))}{(\epsilon + \epsilon^{3/2} - \epsilon^2)} \leqslant e_\epsilon\big(g_K - \big(\epsilon + \epsilon^{3/2}\big)\big) \quad \text{on } D\big(\epsilon + \epsilon^{3/2}\big) \setminus D\big(\epsilon^2\big),$$

where $e_\epsilon = \frac{\epsilon - \epsilon^2 + \beta(\epsilon)}{(\epsilon + \epsilon^{3/2} - \epsilon^2)}$ tends to 1 when ϵ tends to 0. By applying again Lemma 2.5, we deduce that

$$\int\limits_{D(\epsilon+\epsilon^{3/2})} \big(dd^c v_1\big)^n \geqslant e_\epsilon^n (2\pi)^n;$$

which completes the proof of Lemma 2.6. □

2.3. Proof of Corollary 2

If K is a compact set in $\mathbb{C}^n$, we denote by K_ϵ the compact set defined by:

$$K_\epsilon = \big\{z \in \mathbb{C}^n\colon \operatorname{dist}(z, K) \leqslant \epsilon\big\}.$$

Then it is well known that $V_{K_\epsilon} = V_{\hat{K}_\epsilon}$ is continuous ($\hat{K}_\epsilon$ is the polynomially convex hull of K_ϵ), i.e. K_ϵ is $\mathcal{L}$-regular, and $\lim_{\epsilon\to 0} V_{K_\epsilon} = V_K$. In this case $g_{K_\epsilon} = V_{K_\epsilon}$ (see [17]).

If K is a polynomially convex compact subset in $\mathbb{C}^n$, K_ϵ is $\mathcal{L}$-regular and is not necessary polynomially convex. On the other hand, $\hat{K}_\epsilon$ is polynomially convex and $\mathcal{L}$-regular. $g_{K_\epsilon} = V_{K_\epsilon} = V_{\hat{K}_\epsilon}$ is a continuous function in $\mathbb{C}^n$ (g_{K_ϵ} is upper semi-continuous and V_{K_ϵ} is always lower semi-continuous).

If K is a compact set in $\mathbb{C}^n$, we have $\bigcap_{\epsilon>0} \hat{K}_\epsilon = \hat{K}$. Indeed, let us denote by K_0 the compact set $\bigcap_{\epsilon>0} \hat{K}_\epsilon$. We see easily that K_0 is polynomially convex. $K \subset K_\epsilon$, $\hat{K} \subset \hat{K}_\epsilon$ and $\hat{K} \subset K_0$. Conversely, if $z \notin \hat{K}$, there exist a polynomial p and a real number $\epsilon > 0$ such that $|p(z)| > \|p\|_{K_\epsilon} = \|p\|_{\hat{K}_\epsilon}$. We deduce that $z \notin \hat{K}_\epsilon$, $\mathbb{C}^n \setminus \hat{K} \subset \bigcup_{\epsilon>0}(\mathbb{C}^n \setminus \hat{K}_\epsilon)$ and finally $\hat{K} \supset K_0$.

Proof of Corollary 2. K is a polynomially convex compact subset in $\mathbb{C}^n$ and $\mathcal{U}$ is an open neighborhood of K. Then there exist two positive real numbers ϵ and δ such that $\hat{K}_\epsilon \subset \{z \in \mathbb{C}^n\colon g_{\hat{K}_\epsilon}(z) < \delta + \delta^2\} \subset \mathcal{U}$ and for any $0 < \epsilon' \leqslant \epsilon$ and any $0 < \delta' \leqslant \delta$,

$$\hat{K}_{\epsilon'} \subset \big\{z \in \mathbb{C}^n\colon g_{\hat{K}_{\epsilon'}}(z) < \delta' + \delta'^2\big\} \subset \mathcal{U}.$$

If we apply Theorem 1 to the $\mathcal{L}$-regular and polynomially convex compact set $\hat{K}_{\epsilon'}$, there exist an integer $d \geqslant 1$ and n polynomials $p_1, \ldots, p_n$ of degree d such that $\|p_j\|_{\hat{K}_{\epsilon'}} \leqslant 1$ for $\leqslant j \leqslant n$, and

$$\hat{K}_{\epsilon'} \subset \mathcal{P}_{\epsilon',\delta'} \subset \big\{z \in \mathbb{C}^n\colon g_{\hat{K}_{\epsilon'}}(z) < \delta' + \delta'^2\big\} \subset \mathcal{U}.$$

$\mathcal{P}_{\epsilon',\delta'}$ is a special polynomial polyhedron. Precisely, it is the finite union of the connected components of the open set,

$$\Big\{z \in \mathbb{C}^n\colon \sup_{1\leqslant l\leqslant n} \frac{1}{d} \log\big|p_l(z)\big| < \delta' + \beta(\delta')\Big\},$$

that meet $\overline{\{z \in \mathbb{C}^n\colon g_{\hat{K}_{\epsilon'}}(z) < \delta'\}}$, where $0 < \beta(\delta') \leqslant \delta'^2/2$.

The polynomial mapping $F := (p_1, \ldots, p_n)$ of degree d (which depends on ϵ' and δ') can be chosen such that:

(i) F has an homogeneous component $H := (\hat{p}_1, \ldots, \hat{p}_n)$ of degree d, where $\hat{p}_l$ for $1 \leqslant l \leqslant n$ is the homogeneous component of degree d of the polynomial p_l, such that $H^{-1}(\{O\})$ is reduced to $\{O\}$,
(ii) F has a finite set Z of zeros and each zero is of multiplicity one,
(iii) we have:

$$\frac{\operatorname{card}(Z \cap \mathcal{P}_{\epsilon',\delta'})}{d^n} \to 1 \quad \text{when } \delta' \to 0.$$

The proof of Corollary 2 is complete. □

3. Proof of Theorem 3

Let K be a balanced polynomially convex and $\mathcal{L}$-regular compact subset in $\mathbb{C}^n$. In this context, Theorem 2.1 is replaced by the more precise following version.

The pluricomplex Green function g_K is continuous and logarithmically homogeneous (i.e. $g_K(\lambda z) = \log|\lambda| + g_K(z)$, for $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^n$).

Corollary 3.1. *For any $\epsilon > 0$, there exist two integers $d = d(\epsilon) \geqslant 1$ and $N = N(\epsilon) \geqslant 1$ and there exist N homogeneous polynomials $p_1, \ldots, p_N$ of degree d such that $\|p_l\|_K \leqslant 1$ for $1 \leqslant l \leqslant N$, and*

$$g_K(z) - \epsilon \leqslant \sup_{1 \leqslant l \leqslant N} \frac{1}{d} \log|p_l(z)| \leqslant g_K(z) \quad \text{in } \mathbb{C}^n.$$

The proof of this corollary is contained in the proof of Theorem 2.1.

If we apply this corollary for $\epsilon^2/2$ and if we fix ϵ, $d = d_N$, N and the N homogeneous polynomials $p_1, \ldots, p_N$ which appear above, we denote by v_N the following continuous plurisubharmonic function in $\mathbb{C}^n$, logarithmically homogeneous,

$$v_N(z) = \sup_{1 \leqslant l \leqslant N} \frac{1}{d} \log|p_l(z)|.$$

For any $r \in \mathbb{R}$, its open sublevel set is:

$$P_N(r) = \{z \in \mathbb{C}^n \colon v_N(z) < r\}.$$

We have the following inclusions $D(\delta - \epsilon^2/2) \subset P_N(\delta - \epsilon^2/2) \subset D(\delta)$ for any $\delta \in \mathbb{R}$, where $P_N(\delta - \epsilon^2/2)$ is always connected. In particular for $\delta = \epsilon + \epsilon^2$, we obtain:

$$D(\epsilon + \epsilon^2/2) \subset P_N(\epsilon + \epsilon^2/2) \subset D(\epsilon + \epsilon^2).$$

This replaces notation (3) and (4) and inclusions (5) in Section 2.1.

3.1. Proof of the first part

ϵ and $\delta > 0$ are fixed. If we look at how has been proved the first part of Theorem 1, we see that the n polynomials which appear at the end of the process $\mathcal{R}_{n+1}$ and which define the special polyhedron $\mathcal{P}_\epsilon$ are in this context, of the following form:

$$p_l(z) = p_{l,d}(z) + \cdots + p_{l,d'}, \quad \text{for } 1 \leqslant l \leqslant n,$$

where $\frac{d'}{d} = \frac{\alpha}{\alpha+1}$. Of course this last term is greater or equal to $1 - \delta$ for α sufficiently large. Then the first part of Theorem 3 is proved.

3.2. Proof of the second part

Until the end of this paragraph, we suppose that the integer N, which appears in Corollary 3.1 for $\epsilon^2/2$, is greater than $n+1$.

The proof of the second part of Theorem 3, consists to apply a process $(\mathcal{P}_q^H)$ and then to apply a process $(\mathcal{R}_q^H)$ (very similar to process $(\mathcal{R}_q)$ in the proof of Theorem 1), successively for $q = N, N-1, \dots, n+2$.

Process $(\mathcal{P}_q^H)$ consists in modifying a homogeneous polynomial mapping $p = (p_1, \dots, p_q)$ slightly so that the new homogeneous polynomial mapping $\tilde{p} = (\tilde{p}_1, \dots, \tilde{p}_q)$ satisfies: the rational mapping $(\tilde{p}_1/\tilde{p}_q, \dots, \tilde{p}_{q-1}/\tilde{p}_q) : \mathbb{C}^n \setminus \{\tilde{p}_q = 0\} \to \mathbb{C}^{q-1}$ has a range disjoint from the torus $\{(a_1, \dots, a_{q-1}) \in \mathbb{C}^{q-1}\colon |a_j| = 1, 1 \leqslant j \leqslant q-1\}$.

This property is sufficient to apply process $(\mathcal{R}_q^H)$ below.

To apply process $(\mathcal{R}_q^H)$, we suppose that we have a homogeneous polynomial mapping $\tilde{p} = (\tilde{p}_1, \dots, \tilde{p}_q)$ such that the rational mapping $(\tilde{p}_1/\tilde{p}_q, \dots, \tilde{p}_{q-1}/\tilde{p}_q) : \mathbb{C}^n \setminus \{\tilde{p}_q = 0\} \to \mathbb{C}^{q-1}$ has a range disjoint from the torus $\{(a_1, \dots, a_{q-1}) \in \mathbb{C}^{q-1}\colon |a_j| = 1, 1 \leqslant j \leqslant q-1\}$ and such that $\tilde{p}$ defines a polynomial polyhedron of type q which approximate K externally. Then process $(\mathcal{R}_q^H)$ consists in choosing an integer ν sufficiently large such that the homogeneous polynomial mapping $g = (g_1, \dots, g_{q-1})$ in $\mathbb{C}^n$, where $g_j = (\tilde{p}_j)^\nu - (\tilde{p}_q)^\nu$, defines a polynomial polyhedron of type $q-1$ which again approximates K externally.

After $N-n-1$ such constructions, one obtains a polynomial polyhedron of type $n+1$, defined by homogeneous polynomials, and which verifies the inclusions of the second part of Theorem 3, with $\beta(\epsilon) = \epsilon^2/2^{3(N-n)-2}$.

Process $(\mathcal{P}_q^H)$ for $N \geqslant q \geqslant n+2$

Let suppose that we have q homogeneous polynomials p_l, $1 \leqslant l \leqslant q$, of degree $d_q \geqslant 1$, such that $\|p_l\|_K \leqslant 1$ for any $1 \leqslant l \leqslant q$. We suppose also that the polynomial p_q is not identically zero in $\mathbb{C}^n$. So $X_q = \{z \in \mathbb{C}^n\colon p_q(z) = 0\}$ is a proper algebraic subvariety in $\mathbb{C}^n$. Let denote by v_q the continuous plurisubharmonic function in $\mathbb{C}^n$ given by,

$$v_q = \sup_{1 \leqslant l \leqslant q} \frac{1}{d_q} \log |p_l|,$$

and for any $r \in \mathbb{R}$, its open sublevel set:

$$P_q(r) = \{z \in \mathbb{C}^n\colon v_q(z) < r\}.$$

We have $v_q \leqslant g_K$ in $\mathbb{C}^n$. And let suppose in addition that we have the following inclusions:

$$K \subset \overline{D(\epsilon)} \subset P_q\big(\epsilon + \epsilon^2/2^{1+3(N-q)}\big) \subset D\big(\epsilon + \epsilon^2\big). \tag{11}$$

All the homogeneous polynomials p_l are of same degree d_q. Let $(a_1, \dots, a_{n+1}) \in \mathbb{C}^{n+1}$ be such that $|a_j| = 1$ $(1 \leqslant j \leqslant n+1)$. We have $n+1$ homogeneous polynomial mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$,

$$\begin{gathered}(p_1 + a_1 p_q, \dots, p_n + a_n p_q),\\ (p_1 + a_1 p_q, \dots, p_{n-1} + a_{n-1} p_q, p_{n+1} + a_{n+1} p_q),\\ \dots\\ (p_2 + a_2 p_q, \dots, p_{n+1} + a_{n+1} p_q).\end{gathered}$$

Then there exists a homogeneous polynomial mapping $(\tilde{p}_1, \dots, \tilde{p}_{n+1})$ of degree d_q near from $(p_1, \dots, p_{n+1})$ such that:

(i) for any $(a_1, \dots, a_{n+1}) \in \mathbb{C}^{n+1}$ such that $|a_j| = 1$, there is one of these $n+1$ mappings which is in $E \setminus \Sigma$. Indeed, the torus $\{(a_1, \dots, a_{n+1}) \in \mathbb{C}^{n+1}\colon |a_j| = 1, 1 \leqslant j \leqslant n+1\}$ has a real dimension $n+1$, and "the $n+1$ mappings are in Σ" corresponds to $n+1$ complex conditions (or $2n+2$ real conditions).
(ii) $\|\tilde{p}_l\|_K \leqslant 1$ for any $1 \leqslant l \leqslant n+1$.

If $\tilde{p}_l = p_l$ for $n+2 \leqslant l \leqslant q$ and if we denote by $\tilde{v}_q$ the continuous plurisubharmonic function in $\mathbb{C}^n$ given by:

$$\tilde{v}_q = \sup_{1 \leqslant l \leqslant q} \frac{1}{d_q} \log |\tilde{p}_l|,$$

and for any $r \in \mathbb{R}$, its open sublevel set,

$$\tilde{P}_q(r) = \left\{z \in \mathbb{C}^n \colon \tilde{v}_q(z) < r\right\},$$

then we have:

$$K \subset \overline{D(\epsilon)} \subset \tilde{P}_q\left(\epsilon + \epsilon^2/2^{2+3(N-q)}\right) \subset D\left(\epsilon + \epsilon^2\right). \tag{12}$$

Process $(\mathcal{R}_q^H)$ for $N \geqslant q \geqslant n+2$

Let suppose that we have $\tilde{v}_q$ a continuous plurisubharmonic function in $\mathbb{C}^n$ defined as above, where $\tilde{p}_l$ are q homogeneous polynomials of degree d_q such that

(i) $\|\tilde{p}_l\|_K \leqslant 1$ for any $1 \leqslant l \leqslant q$,
(ii) the following rational mapping:

$$\left(\frac{\tilde{p}_1}{\tilde{p}_q}, \ldots, \frac{\tilde{p}_{q-1}}{\tilde{p}_q}\right) \colon \mathbb{C}^n \setminus \{\tilde{p}_q = 0\} \to \mathbb{C}^{q-1},$$

has a range which does not intersect the torus $\{(a_1, \ldots, a_{q-1}) \in \mathbb{C}^{q-1} \colon |a_j| = 1,\ 1 \leqslant j \leqslant q-1\}$. This property is weaker than to ask that the rational mapping is locally finite. But this property is sufficient to apply process $(\mathcal{R}_q^H)$.

We suppose also that we have inclusions (12).

Now, for any $1 \leqslant l \leqslant q-1$ and for any $\nu \in \mathbb{N}^*$, let define the following homogeneous polynomial mapping,

$$F_{l,\nu} = \tilde{p}_l^\nu - \tilde{p}_q^\nu.$$

And for any $r \in \mathbb{R}$, we define the open connected set $P_{q-1}^\nu(r)$ by:

$$P_{q-1}^\nu(r) = \left\{z \in \mathbb{C}^n \colon w_{q-1}^\nu(z) < r\right\},$$

where $w_{q-1}^\nu = \sup_{1 \leqslant l \leqslant q-1} \frac{1}{d_q \nu} \log|F_{l,\nu}|$ is a continuous plurisubharmonic function in $\mathbb{C}^n$.

The following proposition is almost identical to Proposition 2.3. Here the open sets $P_{q-1}^\nu(\epsilon + \alpha_2(\epsilon))$ and $\tilde{P}_q(\epsilon + \alpha_1(\epsilon))$ are connected.

Proposition 3.2. *For any $\epsilon > 0$ sufficiently small, if $\alpha_1(\epsilon) = \epsilon^2/2^{2+3(N-q)}$ and $\alpha_2(\epsilon) = \epsilon^2/2^{3(N-q+1)}$, there exists an integer $\nu_q \geqslant 1$ such that for any $\nu \geqslant \nu_q$, we have*:

$$K \subset \overline{D(\epsilon)} \subset P_{q-1}^\nu\left(\epsilon + \alpha_2(\epsilon)\right) \subset \tilde{P}_q\left(\epsilon + \alpha_1(\epsilon)\right) \subset D\left(\epsilon + \epsilon^2\right). \tag{13}$$

Process $(\mathcal{R}_q^H)$ consists in applying the above proposition to the $q-1$ homogeneous polynomials $F_{l,\nu}$.

Then we choose ν sufficiently large such that $\log 2/(d_q \nu) \leqslant \epsilon^2/2^{3(N-q)+4}$. We define $q-1$ new homogeneous polynomials p_l by $F_{l,\nu}/2$. $\|p_l\|_K \leqslant 1$ for any $1 \leqslant l \leqslant q-1$. We also denote by $P_{q-1}(r)$ the new open sublevel set,

$$P_{q-1}(r) = \left\{z \in \mathbb{C}^n \colon v_{q-1}(z) < r\right\},$$

where v_{q-1} is the plurisubharmonic function in $\mathbb{C}^n$ defined by,

$$v_{q-1} = \sup_{1 \leqslant l \leqslant q-1} \frac{1}{d_q \nu} \log|p_l|.$$

Remark that $v_{q-1} \leqslant g_K$ in $\mathbb{C}^n$. Let denote by $d_{q-1} = d_q \nu$. Then we obtain inclusions (11) with $q-1$ and we are ready to apply process $(\mathcal{P}_{q-1}^H)$.

4. Proofs of Corollaries 4, 5 and 6

4.1. Proof of Corollary 4

We fix $\epsilon > 0$ and we apply Theorem 1 to the compact set K. For simplification, we note $v = v_\epsilon$, defined in (10), and $\mathcal{P} = \mathcal{P}_\epsilon$. Let $\mathcal{F}$ be the set of plurisubharmonic functions in $\mathbb{C}^n$ defined by:

$$\mathcal{F} = \{\varphi \in \mathcal{L}\colon\ \varphi \leqslant v \text{ on } \partial\mathcal{P},\ \varphi(z) \leqslant v(z) + \emptyset(1) \text{ when } z \to \alpha_j,\ 1 \leqslant j \leqslant k'\}.$$

And let w be the sup-envelope $\sup\{\varphi\colon\ \varphi \in \mathcal{F}\}$.

It is relatively easy to verify that this function w is a good candidate for Corollary 4.

(i) If $\varphi \in \mathcal{F}$, $\varphi(z) - (\epsilon + \beta(\epsilon)) \leqslant 0$ on $\mathcal{P}$ and in particular on K. In addition $\varphi(z) - (\epsilon + \beta(\epsilon)) \in \mathcal{L}$. Then

$$w - (\epsilon + \beta(\epsilon)) \leqslant V_K \leqslant g_K \quad \text{on } \mathbb{C}^n,$$

and $w^* \in \mathcal{L}$.

(ii) $\forall \varphi \in \mathcal{F}$, $\varphi(z) \leqslant v(z) = \epsilon + \beta(\epsilon)$ on $\partial\mathcal{P}$ and $\varphi(z) \leqslant v(z) + \emptyset(1)$ when $z \to \alpha_j$, $1 \leqslant j \leqslant k'$. v is maximal on $\mathcal{P} \setminus \{\alpha_j,\ 1 \leqslant j \leqslant k'\}$. Consequently, $\varphi(z) \leqslant v(z)$ on $\mathcal{P}$ and $w(z) \leqslant v(z)$ on the closure of $\mathcal{P}$.

$v \in \mathcal{F}$ and $v \leqslant w \leqslant w^*$ on $\mathbb{C}^n$. Then

$$w(z) = v(z) \quad \text{on the closure of } \mathcal{P},$$

and $w(z) = w^*(z) = v(z)$ on $\mathcal{P}$.

(iii) Let us prove that $w^* = v$ on $\partial\mathcal{P}$.

$\mathcal{P}$ is a bounded open set in $\mathbb{C}^n$ and any connected component ω of $\mathcal{P}$ is a bounded hyperconvex domain in $\mathbb{C}^n$. $v \in \mathcal{C}(\partial\omega)$ and it is evidently continuously extended to a plurisubharmonic function (which is v here) on ω. Then there exists a solution for the following Dirichlet problem for the Monge–Ampère operator (Bedford and Taylor [8] and Blocki [7]): φ_0 is a plurisubharmonic function on $\mathcal{P}$, continuous on the closure of $\mathcal{P}$ such that $\varphi_0(z) = v(z) = \epsilon + \beta(\epsilon)$ on $\partial\mathcal{P}$ and $(dd^c\varphi_0)^n = 0$ on $\mathcal{P}$.

Then if $\varphi \in \mathcal{F}$, $\varphi(z) \leqslant \varphi_0(z)$ on the closure of $\mathcal{P}$ and in particular, $w^*(z) \leqslant \varphi_0(z)$ on $\mathcal{P}$

$$\forall z \in \partial\mathcal{P}, \quad w^*(z) = \limsup_{\xi \in \mathcal{P} \to z} w^*(\xi) \leqslant \varphi_0(z) = v(z).$$

Consequently,

$$w^*(z) = v(z), \quad \forall z \in \partial\mathcal{P}.$$

(iv) Thus, we can deduce that $w^* \in \mathcal{F}$, $w^* \equiv w$ on $\mathbb{C}^n$ and

$$w \in \mathcal{C}(\mathbb{C}^n, [-\infty, +\infty[).$$

(v) Let denote by φ_1 the continuous plurisubharmonic function on $\mathbb{C}^n$ (we recall that $\mathcal{P} \subset D(\epsilon + \epsilon^2)$) defined by:

$$\varphi_1(z) = \begin{cases} v(z) & \text{on } \mathcal{P}, \\ \max\{v(z), g_K(z) - \epsilon^2 + \beta(\epsilon)\} & \text{on } \mathbb{C}^n \setminus \mathcal{P}. \end{cases}$$

We remark that $\varphi_1 \in \mathcal{F} \cap \mathcal{L}^+$, $\varphi_1 \leqslant w$ on $\mathbb{C}^n$ and in particular, $w \in L^\infty_{\mathrm{loc}}(\mathbb{C}^n \setminus Z') \cap \mathcal{L}^+$, and

$$w(z) \geqslant g_K(z) - \epsilon^2 + \beta(\epsilon) \quad \text{on } \mathbb{C}^n \setminus \mathcal{P}.$$

(vi) Let us prove now that w is maximal on $\mathbb{C}^n \setminus (\partial\mathcal{P} \cup Z')$.

Let G be a bounded open subset of $\mathbb{C}^n \setminus (\partial\mathcal{P} \cup Z')$ and let $h \in PSH(G)$ be such that $\limsup_{\xi \in G \to z} h(\xi) \leqslant w(z)$ for all $z \in \partial G$. Let φ_2 be the following plurisubharmonic function in $\mathbb{C}^n$ defined by:

$$\varphi_2(z) = \begin{cases} w(z) & \text{on } \mathbb{C}^n \setminus G, \\ \max\{w(z), h(z)\} & \text{on } G. \end{cases}$$

$\varphi_2 \in \mathcal{F}$, then $\varphi_2 \leqslant w$ in $\mathbb{C}^n$ and in particular $h \leqslant w$ on G. This concludes the proof of the maximality.

(vii) Since $w \in \mathcal{L}^+$, $\int_{\mathbb{C}^n} (dd^c w)^n = (2\pi)^n$ (see [17], p. 212).

(viii) $\int_{\mathbb{C}^n} (dd^c w)^n = (2\pi)^n = \int_{\partial\mathcal{P}} (dd^c w)^n + \int_{Z'} (dd^c w)^n$, where $\int_{Z'} (dd^c w)^n = \int_{Z'} (dd^c v)^n \to (2\pi)^n$. Thus $\int_{\partial\mathcal{P}} (dd^c w)^n \to 0$ when $\epsilon \to 0$.

4.2. Proof of Corollary 5

Let us prove the first part. Let u be a continuous plurisubharmonic function in $\mathcal{L}^+$, L be a compact set in $\mathbb{C}^n$ and ϵ be a positive real number.

If $\lim_{\lambda\in\mathbb{C},|\lambda|\to\infty}(u(\lambda z)-\log|\lambda|)$ exists for any $z\in\mathbb{C}^n\setminus\{O\}$, we do not need to modify u in $\tilde u$ as follows.

Let suppose that it is not the case. Then, for $R>0$ such that $L\subset B(O,R)$, we choose a constant M such that $u(z)\geqslant M(1+\epsilon)$ on $\partial B(O,R)$ and we construct a plurisubharmonic and continuous function $\tilde u$ in $\mathbb{C}^n$ as: $\tilde u(z)=\frac{u(z)}{1+\epsilon}$ if $z\in B(O,R)$ and $\tilde u(z)=\max\{\frac{u(z)}{1+\epsilon},\log(\|z\|/R)+M\}$ if $z\in\mathbb{C}^n\setminus B(O,R)$; $\tilde u$ is in $\mathcal{L}^+$ and satisfies: $\lim_{\lambda\in\mathbb{C},|\lambda|\to\infty}(\tilde u(\lambda z)-\log|\lambda|)$ exists for any $z\in\mathbb{C}^n\setminus\{O\}$.

Let define the function h in $\mathbb{C}^{n+1}$ (as in Theorem 2.1) by:

$$h(\lambda,z)=\begin{cases}|\lambda|\exp\tilde u(\lambda^{-1}z), & \text{if } \lambda\in\mathbb{C}\setminus\{0\},\ z\in\mathbb{C}^n,\\ \lim_{\zeta\to 0,\zeta\neq 0}|\zeta|\exp\tilde u(\zeta^{-1}z), & \text{if } \lambda=0\in\mathbb{C},\ z\in\mathbb{C}^n.\end{cases}$$

Then h is plurisubharmonic, continuous in $\mathbb{C}^{n+1}$, non-negative homogeneous and $\exp(\tilde u(z))=h(1,z)$ for all $z\in\mathbb{C}^n$. In addition $h^{-1}(0)=\{O\}$.

Let $K=\{(\lambda,z)\in\mathbb{C}^{n+1}\colon\ h(\lambda,z)\leqslant 1\}$. K is a balanced polynomially convex and $\mathcal{L}$-regular compact subset in $\mathbb{C}^{n+1}$. The pluricomplex Green function $g_K=\log^+ h$ in $\mathbb{C}^{n+1}$.

According to the second part of Theorem 3, for any $\epsilon>0$, there exist $n+2$ homogeneous polynomials q_l of degree d ($\geqslant 1$) in $\mathbb{C}^{n+1}$ such that

$$\sup_{1\leqslant l\leqslant n+2}\frac{1}{d}\log\bigl|q_l(\lambda,z)\bigr|\leqslant\log h(\lambda,z)\leqslant\sup_{1\leqslant l\leqslant n+2}\frac{1}{d}\log\bigl|q_l(\lambda,z)\bigr|+\epsilon,\quad \text{in } \mathbb{C}^{n+1}.$$

Indeed, $\sup_{1\leqslant l\leqslant n+2}\frac{1}{d}\log|q_l(\lambda,z)|$ is in $\mathcal{L}^+$, is maximal outside the origin because it is log-homogeneous, and $\sup_{1\leqslant l\leqslant n+2}\frac{1}{d}\log^+|q_l(\lambda,z)|=g_{K_\epsilon}(\lambda,z)$ in $\mathbb{C}^{n+1}$, where $K_\epsilon=\{(\lambda,z)\in\mathbb{C}^{n+1}\colon\ \sup_{1\leqslant l\leqslant n+2}|q_l(\lambda,z)|\leqslant 1\}$.

Consequently, in $L\subset B(O,R)\subset\mathbb{C}^n$, we have:

$$\sup_{1\leqslant l\leqslant n+2}\frac{1}{d}\log\bigl|q_l(1,z)\bigr|\leqslant\log h(1,z)=\frac{u(z)}{1+\epsilon}\leqslant\sup_{1\leqslant l\leqslant n+2}\frac{1}{d}\log\bigl|q_l(1,z)\bigr|+\epsilon.$$

We achieve the proof of the first part by using the uniform continuity of u in L.

The second part of Corollary 5 is a direct consequence of Theorem 3 and of the previous arguments.

4.3. Proof of Corollary 6

Corollary 6 is a consequence of Weil's integral formula [11] and of Corollary 2.

Indeed let K be a polynomially convex compact subset of $\mathbb{C}^n$ and let f be an holomorphic function in an open neighborhood $\mathcal{U}$ of K.

According to Corollary 2, there exists a proper polynomial mapping $F=(p_1,\dots,p_n)$ of degree d such that

$$K\subset\mathcal{P}\subset\overline{\mathcal{P}}\subset\mathcal{U},$$

where the special polynomial polyhedron $\mathcal{P}$ is a finite union of connected components $\mathcal{P}_\alpha$ of the open neighborhood $\{z\in\mathbb{C}^n\colon\ \sup_{1\leqslant l\leqslant n}|p_l(z)|<1\}$ of K. We denote by σ_α the n-dimensional part of the boundary of each $\mathcal{P}_\alpha$, with the classical orientation.

Weil's integral formula applied to $\mathcal{P}_\alpha$ gives us:

$$f(z)=\frac{1}{(2i\pi)^n}\int_{\sigma_\alpha}\frac{\delta(\zeta,z)f(\zeta)\,d\zeta_1\dots d\zeta_n}{\prod_{i=1}^n[p_i(\zeta)-p_i(z)]},\quad z\in\mathcal{P}_\alpha;$$

$\delta(\zeta,z)$ is the determinant

$$\delta(\zeta,z)=\det[P_{il}]_{i,l=1,\dots,n},$$

while P_{il} is defined by the relations:

$$p_i(\zeta)-p_i(z)\equiv\sum_{l=1}^n(\zeta_l-z_l)P_{il}(\zeta,z),\quad i=1,\dots,n.$$

Thus, $P_{il}(\zeta, z)$ are polynomials of degree $\leqslant d$, both in $z = (z_1, \ldots, z_n)$ and in $\zeta = (\zeta_1, \ldots, \zeta_n)$.

And Weil's integral formula is again valid for $\mathcal{P}$, i.e. we have:

$$f(z) = \frac{1}{(2i\pi)^n} \sum_\alpha \int_{\sigma_\alpha} \frac{\delta(\zeta, z) f(\zeta)\, d\zeta_1 \ldots d\zeta_n}{\prod_{i=1}^n [p_i(\zeta) - p_i(z)]}, \quad z \in \mathcal{P}.$$

Indeed for $z \in \mathcal{P}_\alpha$, $\int_{\sigma_{\alpha'}} \frac{\delta(\zeta,z) f(\zeta)\, d\zeta_1 \ldots d\zeta_n}{\prod_{i=1}^n [p_i(\zeta) - p_i(z)]} = 0$ for any $\alpha' \neq \alpha$, because Weil's integral formula is a consequence of Martinelli–Bochner's integral formula where the situation is the same. Then we have:

$$f(z) = \sum_\alpha \sum_{k_1, \ldots, k_n = 0}^{\infty} P_{\alpha k_1 \ldots k_n}(z) p_1(z)^{k_1} \ldots p_n(z)^{k_n} = \sum_{k_1, \ldots, k_n = 0}^{\infty} Q_{k_1 \ldots k_n}(z) \quad z \in \mathcal{P},$$

where

$$P_{\alpha k_1 \ldots k_n}(z) = \frac{1}{(2i\pi)^n} \int_{\sigma_\alpha} \frac{\delta(\zeta, z) f(\zeta)\, d\zeta_1 \ldots d\zeta_n}{[p_1(\zeta)]^{k_1+1} \ldots [p_n(\zeta)]^{k_n+1}},$$

and

$$Q_{k_1 \ldots k_n}(z) = \sum_\alpha P_{\alpha k_1 \ldots k_n}(z) p_1(z)^{k_1} \ldots p_n(z)^{k_n},$$

is a complex polynomial of degree $\leqslant n.d + d(k_1 + \cdots + k_n)$. The proof of Corollary 6 is complete.

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